Gibbs measures: idea and existence

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A measure μ on X is called *quasi-invariant* if $T^*\mu \ll \mu$, i.e. $\mu(E)=0 \Rightarrow \mu(T^{-1}(E))=0$.

It has a Jacobian, defined as $J_\mu^{-1}(x) = \frac{dT^*\mu}{d\mu} = \lim_{n\to\infty} \frac{\mu C(x_2,\dots,x_n)}{\mu C(x_1,\dots,x_n)}$.

Then $\int v\,d\mu = \int \sum_{Ty=x} v(y)\frac{1}{J_\mu(y)}\,d\mu = \int L_\varphi(v)\,d\mu$, where $\varphi := -\log J_\mu$. $= \int v\, dL_\varphi^*\mu$.

So, if $-\log J_\mu$ is Hölder, μ - is the unique L_φ^* invariant measure.

Def Let $\varphi \in C(X)$. A probability measure μ is called *Gibbs measure* w.r.t. φ if for some $A, B>0$, $C\in\mathbb{R}$, we have

$$A \le \frac{\mu(C(x_1,\dots,x_n))}{\exp\left((S_n\varphi)_{C(x_1,\dots,x_n)} + Cn\right)} \le B.$$

In particular, $-\log J_\mu(x) = \varphi(x)$.

Observe that $1 = \sum \mu(C(x_1,\dots,x_n)) \asymp \sum_{C\in\xi_n} \exp((S_n\varphi)_C \cdot e^{Cn}$.

Thus $\sum (\exp S_n\varphi)_C \asymp e^{-Cn}$. Take log, divide by n, to obtain $\boxed{P(\varphi) = -C}$

Let us start with the normalized case: $\sum_{T(y)=x} e^{\varphi(y)} = 1$, i.e. $L_\varphi(1)=1$.

Lemma. Let $\varphi \in B_0$, ν: the unique probability measure with $L_\varphi^*\nu = \nu$. Then

1) ν is T-invariant.

2) $e^{-\omega_n(\varphi)} \le \frac{\nu(C(x_1,\dots,x_n))e^{-\varphi(x)}}{\nu(x_2,\dots,x_n)} \le e^{\omega_n(\varphi)}$ for $\forall x=(x_1,\dots)$

3) $0 = P(\varphi) = h_\nu(T) + \int_X \varphi\,d\nu$. ν - unique such T-invariant measure.

Remark The lemma implies that ν is φ-Gibbs, with $C=0$, $B = e^{\|\varphi\|_{B_0}} (= \Pi e^{\omega_k(\varphi)})$, $A = e^{-\|\varphi\|_{B_0}}$.

Pf. 1) Observe that $L_\varphi(f\circ T) = \sum_{T(y)=x} e^{\varphi(y)} f(T(y)) = f(x)\sum_{T(y)=x} e^{\varphi(y)} = f(x)$

Thus $\int f\circ T\,d\nu = \int f\circ T\,d(L_\varphi^*\nu) = \int L_\varphi(f\circ T)\,d\nu = \int f\,d\nu$.

2) Note that

$$\int_{C(x_1\dots x_n)} e^{-\varphi}\,d\nu \overset{L_\varphi^*\nu=\nu}{=} \int_X L_\varphi(e^{-\varphi}\chi_{C(x_1\dots x_n)})\,d\nu = \int_X \sum_{T(y)=x} e^{\varphi(y)}\cdot e^{-\varphi(y)}\chi_{C(x_1\dots x_n)}(y)\,d\nu = \int_{C(x_2,\dots x_n)} d\nu$$

Note that $e^{-\omega_n(\varphi)} \le \frac{\int_{C(x_1\dots x_n)} e^{-\varphi}\,d\nu}{\nu(C(x_1,\dots,x_n))e^{-\varphi(x)}} \le e^{\omega_n(\varphi)}$.

3). We know that $P(\varphi) \ge h(\nu) + \int_X \varphi\,d\nu$ for any invariant ν.

Observe that for our ν, $J_\nu(x) = e^{-\varphi(x)}$, so

$$(*) \quad -\sum_{Ty=x} J_\nu^{-1}(y)\log J_\nu(y) + \sum_{Ty=x}\frac{\varphi(y)}{J_\nu(y)} = 0.$$ As in the variation principle, integrate w.r.t. ν to get

$$\int \sum_{Ty=x} J_\nu^{-1}(y)\log J_\nu(y)\,d\nu(x) + \sum_{Ty=x}\int \frac{\varphi(y)}{J_\nu(y)}\,d\nu(x) = \int (L_\varphi \log J_\nu)\,d\nu + \int (L_\varphi\varphi)\,d\nu \overset{L_\varphi^*\nu=\nu}{=}$$
$$\int \log J_\nu\,d\nu + \int \varphi\,d\nu = h(\nu) + \int \varphi\,d\nu.$$

The equality is reached iff there is $\ominus$ in $\otimes$ ν-a.e. By the entropy inequality, it means $J_\nu = e^{-\varphi}$ a.e. ∎

Now let us do the general case. Here, we will take $\varphi \in C^\delta$, so that

$\tilde\varphi = \varphi - \log h\circ T + \log h - \log\beta \in C^\delta$.

Thm. Let $\varphi \in C^\delta$, $L_\varphi h = \beta h$, $L^*_\varphi \mu = \beta\mu$. Then $\nu = h\mu$ is the unique T-invariant measure with $P(\varphi) = h(\nu) + \int_X \varphi\, d\nu$.
μ is also Gibbs for φ with $C = -P(\varphi)$.
So is ν, with $C = -P(\varphi)$.
ν is also ergodic.

Remark The theorem can be proven for $\varphi \in B_1 := \{\sum k\,\omega_k(\varphi) < \infty\}$, which guarantees $\tilde\varphi \in B_0$.

Pf. $h(\nu) + \int_X \tilde\varphi\, d\nu = 0$, thus $h(\nu) + \int_X \varphi\, d\nu = h(\nu) + \int_X \tilde\varphi\, d\nu + \int \log h\circ T\, d\nu - \int \log h\, d\nu + \int P(\varphi)\, d\nu \overset{\nu\text{ invariant}}{=} h(\nu) + \int \tilde\varphi\, d\nu + P(\varphi) = P(\varphi)$, and ν is unique such invariant measure, since it was unique for $\tilde\varphi$.
Observe now that $\mu(C(x_1,\dots,x_n)) = \int_{C(x_1,\dots,x_n)} h\, d\nu \asymp e^{\log h}\nu(C(x_1,\dots,x_n)) \asymp e^{S_n(\tilde\varphi) + \log h} = e^{S_n(\varphi) - n\log\beta}$ ($S_n(\tilde\varphi) = S_n(\varphi) - \log h\circ T^n + \log h - n\log\beta$).
The same is true for ν.
If A is a T-invariant set, then $\nu_0 = \nu|_A$ also satisfies $L_{\tilde\varphi}\nu_0 = \nu_0$. Thus $\nu_0 = \nu$, so $\nu(A) = 1$ or $\nu(A) = 0$.

Corollary. Let μ be quasi-invariant, $\log J_\mu =: -\varphi \in C^\delta$. Then $\exists!\ \nu$-invariant, $\nu \ll \mu$, and $\exists \gamma \in C^\delta$:
$\varphi = -\log J_\nu + \gamma\circ T - \gamma$.

Pf. As we know, $\varphi = -\log J_\mu \iff L^*_\varphi \mu = \mu$. Thus $P(\varphi) = 0$. Let now ν be the ergodic measure for $\tilde\varphi$, $\gamma = \log h$.
If ν_1 is other such measure, and $h_1 = \frac{d\nu_1}{d\mu}$, then the invariance of ν_1 implies $L_\varphi h_1 = h_1$, so $h_1 = h$, $\nu_1 = \nu$ ∎

Def. We say that $\varphi_1 \sim \varphi_2$ (φ_1 is cohomological to φ_2), $\varphi_1, \varphi_2 \in C^\delta$, if $\exists \gamma \in C^\delta$: $\varphi_1(x) = \varphi_2(x) + \gamma(Tx) - \gamma(x)$.

Lemma. 1) $\varphi_1 \sim \varphi_2 \Rightarrow P(\varphi_1) = P(\varphi_2)$
2) φ_1 and φ_2 has the same equilibrium measure if $\varphi_1 \sim \varphi_2 - P(\varphi_2) + P(\varphi_1)$.

Pf. 1) is established by a $S_n\varphi_1 = S_n\varphi_2 + \gamma\circ T^n - \gamma$, so $P(\varphi_1) \le P(\varphi_2) + \lim_{n\to\infty}\frac{2\|\gamma\|_\infty}{n}$. Same for $P(\varphi_2) \le P(\varphi_1)$.
2) Follows from the fact that ν is the equilibrium measure for φ iff $-\log J_\nu \sim \varphi - P(\varphi)$. ∎

Def The equilibrium measure for the potential φ is the T-invariant ν with $P(\varphi) = h(\nu) + \int_X \varphi\, d\nu$.

Thm (Full variational principle)
$\forall \varphi \in C(X)$: $P(\varphi) = \sup\{h(\mu) + \int_X \varphi\, d\mu;\ \mu \in M(X,T)\}$.

Pf. We know $\ge$. We also know $(=)$ on C^δ.
Observe also that $P(\varphi)$ is 1-Lipschitz on $C(X)$. So is RHS. Thus, it is enough to prove it on a dense subset of $C(X)$, i.e. on C^δ ∎

Def $\nu \cong \mu$ (ν is strongly equivalent to μ) if $\exists A, B > 0$: $\forall E \subset X$
$A \le \frac{\mu(E)}{\nu(E)} \le B$.
For example, μ and ν in the previous thm are strongly equivalent, with $A = (\inf h)^{-1}$, $B \ge \sup h$.

Thm. Let μ be a probability measure on X. TFAE:

1. μ is quasi-invariant, and $\log J_\mu \in C^\delta$.
2. μ is Gibbs for some $\varphi \in C^\delta$.
3. $\exists\, \varphi \in C^\delta$, ν - invariant probability measure: $\nu \simeq \mu$ and $P(\varphi) = h(\nu) + \int_X \varphi\, d\nu$.

Pf
1) $\Rightarrow$ 2) By corollary: $\mu \simeq h\nu$ for $-\log J_\mu$ - Gibbs ν.
2) $\Rightarrow$ 1) $\varphi = -\log J_\mu \in C^\delta$.
2) $\Rightarrow$ 3) Let ν be the equilibrium measure for φ. μ and ν are both Gibbs with the same $C = -P(\varphi)$. Thus $\mu \simeq \nu$.
3) $\Rightarrow$ 2) ν is the equilibrium measure for φ, so it is Gibbs and $\mu \simeq \nu$, so it is also Gibbs

Let us consider the case when $\varphi \equiv 0$. Then $P(\varphi) = h_{top}(T)$, and the equilibrium measure ν is the measure of maximal entropy for T. We'll study it in more detail later.